Kohn-Vogelius formulation and high-order topological asymptotic formula for identifying small obstacles in a fluid medium

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Abstract

This paper concerns the identification of a small obstacle immersed in a Stokes flow from boundary measurements. The proposed approach is based on the Kohn-Vogelius formulation and the topological sensitivity analysis method. We derive a high order asymptotic formula describing the variation of a Kohn-Vogelius type functional with respect to the insertion of a small obstacle inside the fluid flow domain. The obtained asymptotic formula will serve as very useful tools for developing accurate and robust numerical reconstruction algorithms.

1 Introduction

The development and application of the nondestructive inspection technique has recently received considerable interest in the engineering as well as in the applied mathematics. Such a technique leads to determine interior information about medium from boundary measurements data.

For this purpose, several approaches and methods have been developed, one can cite: the identification of cracks by boundary measurements [1, 2, 3, 4, 5], the reconstruction of electromagnetic imperfections of a small diameter from measurements on the boundary [6, 7], the determination of small conductivity imperfections inside a conductor have been established in [8, 9, 10], the

reconstruction of imperfections of small diameter in an elastic medium using boundary integral formula [11, 12], the identification of a finite number of interior particles of small diameter related to the full Maxwell equations [13], ... etc.

This paper is concerned with the fluid mechanics area. The objective of this work is to derive a high-order asymptotic formula leading to determine the location, shape and size of an obstacle of a small volume immersed in a fluid medium from boundary measurements.

The proposed method is based on the Kohn-Vogelius formulation [14, 15] and the topological sensitivity analysis method [16, 17, 18, 19, 20, 21, 22, 23]. We derive a high-order topological asymptotic expansion for a Kohn-Vogelius type functional with respect to the presence of a small obstacle inside the fluid flow domain. An asymptotic formula is derived giving the relation between the known boundary data and the unknown obstacle properties; location, size and shape.

The obtained asymptotic formula will serve as very useful tools for developing very effective reconstruction algorithms for identifying small obstacles from boundary measurements. Such algorithms can be applied in various applications like fiber-reinforced polymers [24, 25], colloid [26, 27, 28] and casting or injection filling [29, 30, 31] where the design of the mixing of liquid metallic should be optimized.

The rest of this paper is organized as follows. In Section 2, we present the formulation of the considered problem. We introduce the Kohn-Vogelius concept and the topological sensitivity technique. The obstacle reconstruction problem is formulated as a topology optimization one minimizing a Kohn-Vogelius type functional. Section 3 is devoted to a preliminary estimate describing the variation of the considered Kohn-Vogelius functional with respect to the presence of a small obstacle. In Section 4, we establish an asymptotic expansion for the perturbed velocity and pressure fields. In Section 5, we extend the topological derivative notion for the high-order case. We derive a high-order topological asymptotic formula for the Stokes operator. The paper ends by a conclusion and perspective works.

2   Formulation of the problem

We consider a viscous and incompressible fluid flow in a bounded domain Ω of $\mathbb{R}^3$, with smooth boundary $\partial\Omega$. Let Γ and Σ be two parts of $\partial\Omega$ such that $\overline{\partial\Omega} = \overline{\Gamma} \cup \overline{\Sigma}$ and $\Gamma \cap \Sigma = \emptyset$. We assume that the fluid flow is in a laminar regime, in such way that the convection term can be neglected and the Navier-Stokes equations may be approximated by the non-stationary Stokes system.

Inside the fluid flow domain, we assume the existence of a small obstacle $\mathcal{O}_{z,\varepsilon} = z + \varepsilon\mathcal{O}$ that is characterized by its center z, its size ε and its shape $\mathcal{O}$, with $\mathcal{O}$ is a bounded domain of $\mathbb{R}^3$ containing the origin, whose boundary $\partial\mathcal{O}$ is connected and piecewise of class $\mathcal{C}^1$.

The aim of this work is to develop an efficient approach for identifying the unknown parameters (location, size and shape) of the obstacle $\mathcal{O}_{z,\varepsilon}$ from boundary measurement of the velocity field on a part of the boundary. This issue can be formulated as a geometric inverse problem as follow:

- Given a Neumann and Dirichlet data on the accessible boundary Σ: an imposed force G and a measured velocity field $\mathcal{W}_m$.

- Determine the location z, size ε and shape $\mathcal{O}$ of the unknown obstacle $\mathcal{O}_{z,\varepsilon}$ such that the solution $(w_{\mathcal{O}_{z,\varepsilon}}, q_{\mathcal{O}_{z,\varepsilon}})$ of the Stokes equations satisfies the following over-determined boundary value problem

$$\begin{cases} -\nu\Delta w_{\mathcal{O}_{z,\varepsilon}} + \nabla q_{\mathcal{O}_{z,\varepsilon}} & = F & \text{in} & \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ \operatorname{div} w_{\mathcal{O}_{z,\varepsilon}} & = 0 & \text{in} & \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ \sigma(w_{\mathcal{O}_{z,\varepsilon}}, q_{\mathcal{O}_{z,\varepsilon}})\mathbf{n} & = G & \text{on} & \Sigma, \\ w_{\mathcal{O}_{z,\varepsilon}} & = \mathcal{W}_m & \text{on} & \Sigma, \\ w_{\mathcal{O}_{z,\varepsilon}} & = 0 & \text{on} & \Gamma, \\ w_{\mathcal{O}_{z,\varepsilon}} & = 0 & \text{on} & \partial\mathcal{O}_{z,\varepsilon}. \end{cases}$$

where $w_{\mathcal{O}_{z,\varepsilon}}$ is the velocity field, $q_{\mathcal{O}_{z,\varepsilon}}$ is the pressure field, ν is the kinematic viscosity coefficient of the fluid and F is the gravitational force.

In order to examine the considered geometric inverse problem, we propose in this paper a new approach combining the advantages of the Kohn-Vogelius formulation and the topological sensitivity analysis method.

2.1 Kohn-Vogelius formulation

The first step of our approach is based on the Kohn-Vogelius formulation which rephrase the considered geometric inverse problem into a topology optimization one. It leads to define for any given obstacle $\mathcal{O}_{z,\varepsilon}$ two forward problems: The first one which be called Neumann problem is associated to the Neumann datum F:

$$(\mathcal{P}_\varepsilon^N)\begin{cases} -\nu\Delta w_\varepsilon^N + \nabla q_\varepsilon^N & = G & \text{in} & \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ \operatorname{div} w_\varepsilon^N & = 0 & \text{in} & \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ \sigma(w_\varepsilon^N, q_\varepsilon^N)\mathbf{n} & = F & \text{on} & \Sigma, \\ w_\varepsilon^N & = 0 & \text{on} & \Gamma, \\ w_\varepsilon^N & = 0 & \text{on} & \partial\mathcal{O}_{z,\varepsilon}. \end{cases}$$

The second one is associated to the measured velocity $\mathcal{W}_m$ which will be called as the Dirichlet problem:

$$(\mathcal{P}_\varepsilon^D) \begin{cases} -\nu\Delta w_\varepsilon^D + \nabla q_\varepsilon^D &= G & \text{in} & \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ \text{div } w_\varepsilon^D &= 0 & \text{in} & \Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}, \\ w_\varepsilon^D &= \mathcal{W}_m & \text{on} & \Sigma, \\ w_\varepsilon^D &= 0 & \text{on} & \Gamma, \\ w_\varepsilon^D &= 0 & \text{on} & \partial\mathcal{O}_{z,\varepsilon}. \end{cases}$$

As one can remark here, if $\mathcal{O}_{z,\varepsilon}$ coincides with the exact obstacle $\mathcal{O}_{z^*,\varepsilon^*} = z^* + \varepsilon^*\mathcal{O}^*$, the misfit between the solutions become zero ($w_\varepsilon^N = w_\varepsilon^D$). Starting from this remark, the inverse problem can be formulated as a topology optimization one. The unknown obstacle will be characterized as the minimum of the following Kohn-Vogelius type functional [15]

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}) = \nu \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla w_\varepsilon^N - \nabla w_\varepsilon^D\right|^2 \, dx.$$

More precisely, the identification problem can be formulated as follow:

$$(\mathcal{P}_{opt}) \begin{cases} \text{find } \mathcal{O}_{z^*,\varepsilon^*} \subset \Omega, \text{ such that} \\ \mathcal{K}(\Omega \setminus \overline{\mathcal{O}}_{z^*,\varepsilon^*}) = \min_{\mathcal{O}_{z,\varepsilon}\in\mathcal{D}_{ad}} \mathcal{K}(\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}), \end{cases}$$

where $\mathcal{D}_{ad}$ is a given set of admissible domains.

It is interesting to note that, in the absence of obstacles the function $\mathcal{K}$ reads

$$\mathcal{K}(\Omega) = \nu \int_\Omega \left|\nabla w_0^N - \nabla w_0^D\right|^2 \, dx,$$

where (w_0^N, q_0^N) and (w_0^D, q_0^D) satisfy the Stokes equations in the non perturbed domain

$$(\mathcal{P}_0^N) \begin{cases} -\nu\Delta w_0^N + \nabla q_0^N &= G & \text{in} & \Omega, \\ \text{div } w_0^N &= 0 & \text{in} & \Omega, \\ \sigma(w_0^N, q_0^N)\mathbf{n} &= F & \text{on} & \Sigma, \\ w_\varepsilon^N &= 0 & \text{on} & \Gamma, \end{cases}$$

$$(\mathcal{P}_0^D) \begin{cases} -\nu\Delta w_0^D + \nabla q_0^D &= G & \text{in} & \Omega, \\ \text{div } w_0^D &= 0 & \text{in} & \Omega, \\ w_0^D &= \mathcal{W}_m & \text{on} & \Sigma, \\ w_0^D &= 0 & \text{on} & \Gamma. \end{cases}$$

2.2 Sensitivity analysis method

To solve the topological optimization problem $(\mathcal{P}_{opt})$ and identify the unknown
obstacle, we will propose a simplified approach based on the topological sen-
sitivity analysis for the Kohn-Vogelius function $\mathcal{K}$.

The classical topological sensitivity analysis method is based on a first
order asymptotic expansion of the form

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) = \rho(\varepsilon)\delta\mathcal{K}(z) + o\big(\rho(\varepsilon)\big), \; \forall z \in \Omega,$$

where

- $\varepsilon \mapsto \rho(\varepsilon)$ is a positive scalar function going to zero with ε.

- $z \mapsto \delta\mathcal{K}(z)$ is the first order topological gradient.

In the context of fluid mechanics, the first order topological sensitivity anal-
ysis has been derived for the Stokes operator in [19, 21, 32], the quasi-Stokes
in [22] and for the stationary Navier-Stokes equations in [20]. In [33], Caubet
and Dambrine have used the Kohn-Vogelius formulation combined with the
topological sensitivity analysis to locate obstacles in a fluid flow domain.
In this work, we extend the topological sensitivity notion for the high-order
case. We will derive a high-order topological asymptotic expansion for the
Kohn-Vogelius functional $\mathcal{K}$ with respect to the presence of the Dirichlet ge-
ometric perturbation $\mathcal{O}_{z,\varepsilon}$ inside the domain Ω. It consists in studying the
variation $\mathcal{K}(\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega)$ with respect to ε and establishing an asymptotic
formula on the form

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) = \sum_{q=1}^{m} \rho_q(\varepsilon)\delta\mathcal{K}^q(z, \mathcal{O}) + o(\rho_m(\varepsilon)) \tag{1}$$

where

- $\varepsilon \mapsto \rho_q(\varepsilon)$, $1 \leq q \leq m$ are positive scalar functions defined on $\mathbb{R}^+$
  verifying $\rho_{q+1}(\varepsilon) = o(\rho_q(\varepsilon))$ and $\lim\limits_{\varepsilon \to 0} \rho_q(\varepsilon) = 0$.

- $\delta\mathcal{K}^q$ denotes the qth topological derivative of the functional $\mathcal{K}$.

- $m \in \mathbb{N}^*$ an arbitrary order of the asymptotic expansion.

To this end, we start our analysis by deriving a preliminary result showing
the relationship between the quantity $\mathcal{K}(\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega)$ and the variations
of the Neumann and Dirichlet perturbed solutions $(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)$ and
$(w_\varepsilon^D - w_0^D, q_\varepsilon^D - q_0^D)$.

3   The Kohn-Vogelius functional variation

In this section, we discuss the sensitivity of the considered Kohn-Vogelius functional $\mathcal{K}$ with respect to the presence of a small obstacle $\mathcal{O}_{z,\varepsilon}$ inside the fluid flow domain. We will prove that the variation $\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega)$ can be only described by four integral terms, involving the discrepancy between the initial and the perturbed Stokes problem solutions.

Theorem 3.1. *Let $\mathcal{O}_{z,\varepsilon}$ be a small obstacle, strictly embedded into the fluid flow domain Ω. In the presence of $\mathcal{O}_{z,\varepsilon}$, the Kohn-Vogelius function $\mathcal{K}$ admits the following variation*

$$\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) = \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)\mathbf{n}w_0^N ds$$
$$- \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D, q_\varepsilon^D - q_0^D)\mathbf{n}w_0^D ds \qquad (2)$$
$$+ \nu\int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^D\right|^2 dx - \nu\int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^N\right|^2 dx.$$

Proof. The variation of the function $\mathcal{K}$ is given by

$$\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) = \nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla w_\varepsilon^N - \nabla w_\varepsilon^D\right|^2 dx - \nu\int_{\Omega} \left|\nabla w_0^N - \nabla w_0^D\right|^2 dx$$

$$= \nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla w_\varepsilon^N\right|^2 dx + \nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla w_\varepsilon^D\right|^2 dx - 2\nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \nabla w_\varepsilon^D \nabla w_\varepsilon^N dx$$

$$- \nu\int_{\Omega} \left|\nabla w_0^N\right|^2 dx - \nu\int_{\Omega} \left|\nabla w_0^D\right|^2 dx + 2\nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \nabla w_0^D \nabla w_0^N dx.$$

As one can observe $\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega)$ can be decomposed as

$$\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) = \mathcal{T}_N(\varepsilon) + \mathcal{T}_D(\varepsilon) - 2\mathcal{T}_M(\varepsilon),$$

with $\mathcal{T}_N$ is the Neumann term

$$\mathcal{T}_N(\varepsilon) = \nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla w_\varepsilon^N\right|^2 dx - \nu\int_{\Omega} \left|\nabla w_0^N\right|^2 dx,$$

$\mathcal{T}_D$ is the Dirichlet term

$$\mathcal{T}_D(\varepsilon) = \nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla w_\varepsilon^D\right|^2 dx - \nu\int_{\Omega} \left|\nabla w_0^D\right|^2 dx,$$

and $\mathcal{T}_M$ is the mixed term

$$\mathcal{T}_M(\varepsilon) = \nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \nabla w_\varepsilon^D \nabla w_\varepsilon^N dx - \nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \nabla w_0^D \nabla w_0^N dx.$$

Next, we will examine each term separately. We start our analysis by studying
the term $\mathcal{T}_N$.

$-$*Variation of the Neumann term:* The term $\mathcal{T}_N$ reads

$$
\begin{aligned}
\mathcal{T}_N(\varepsilon) \;&=\; \nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla w_\varepsilon^N\right|^2 dx - \nu \int_{\Omega} \left|\nabla w_0^N\right|^2 dx \\
&=\; \nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \nabla(w_\varepsilon^N - w_0^N)\nabla w_\varepsilon^N dx + \nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \nabla(w_\varepsilon^N - w_0^N)\nabla w_0^N dx \\
&\quad-\; \nu \int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^N\right|^2 dx.
\end{aligned}
$$

Using Green formula, from the Stokes system satisfied by $(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)$,
we obtain

$$
\nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \nabla(w_\varepsilon^N - w_0^N)\nabla w_0^N dx = \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)\mathbf{n} w_0^N ds.
$$

In addition,

$$
\nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \nabla(w_\varepsilon^N - w_0^N)\nabla w_\varepsilon^N dx\, dt = 0.
$$

Then, the term $\mathcal{T}_N$ can be written as

$$
\mathcal{T}_N(\varepsilon) \;=\; \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)\mathbf{n} w_0^N ds - \nu \int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^N\right|^2 dx. \tag{3}
$$

$-$*Variation of the Dirichlet term:* We have

$$
\begin{aligned}
\mathcal{T}_D(\varepsilon) \;&=\; \nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla w_\varepsilon^D\right|^2 dx - \nu \int_{\Omega} \left|\nabla w_0^D\right|^2 dx \\
&=\nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla(w_\varepsilon^D - w_0^D)\right|^2 dx + 2\nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \nabla(w_\varepsilon^D - w_0^D)\nabla w_0^D dx - \nu \int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^D\right|^2 dx.
\end{aligned}
$$

Using Green formula, the system $(\mathcal{P}_0^D)$ implies

$$
\nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \nabla(w_\varepsilon^D - w_0^D)\nabla w_0^D dx = \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx - \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D, q_0^D)\mathbf{n} w_0^D ds.
$$

Then, the Dirichlet term admits the following variation

$$
\begin{aligned}
\mathcal{T}_D(\varepsilon) \;=\;&\; \nu \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} \left|\nabla(w_\varepsilon^D - w_0^D)\right|^2 dx + 2 \int_{\Omega\setminus\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx \\
&-\nu \int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^D\right|^2 dx - 2 \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D, q_0^D)\mathbf{n} w_0^D ds.
\end{aligned}
$$

Using Green formula, from the system verified by $(w_\varepsilon^D - w_0^D, q_\varepsilon^D - q_0^D)$, we deduce

$$
\begin{aligned}
\mathfrak{T}_D(\varepsilon) \;=\;& -\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D, q_\varepsilon^D - q_0^D)\mathbf{n}w_0^D ds + 2\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx \\
& -2\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D,\, q_0^D)\mathbf{n}w_0^D ds - \nu\int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^D\right|^2 dx. \qquad (4)
\end{aligned}
$$

$-$ *Variation of the mixed term:* We have

$$
\mathfrak{T}_M(\varepsilon) = \nu\int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} \nabla w_\varepsilon^N \nabla w_\varepsilon^D dx\, dt - \nu\int_{\Omega} \nabla w_0^N \nabla w_0^D dx.
$$

From the weak formulation of problems $(\mathcal{P}_\varepsilon^N)$ and $(\mathcal{P}_0^N)$, it follows that

$$
\mathfrak{T}_M(\varepsilon) = \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx - \int_{\mathcal{O}_{z,\varepsilon}} F w_0^D dx.
$$

From the fact that $-\nu\Delta w_0^D + \nabla q_0^D = F$ in $\mathcal{O}_{z,\varepsilon}$ and div $w_0^D = 0$ in $\mathcal{O}_{z,\varepsilon}$, one can deduce

$$
\int_{\mathcal{O}_{z,\varepsilon}} F w_0^D dx = \nu\int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^D\right|^2 dx + \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D,\, q_0^D)\mathbf{n}w_0^D ds.
$$

Consequently, the term $\mathfrak{T}_M$ can be written as

$$
\mathfrak{T}_M(\varepsilon) = \int_{\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}} F(w_\varepsilon^D - w_0^D)dx - \nu\int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^D\right|^2 dx - \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_0^D,\, q_0^D)\mathbf{n}w_0^D ds.
$$
$$(5)$$

$-$ *Variation of the Kohn-Vogelius function $\mathcal{K}$:* Combining the variations (3), (4) and (5), one can deduce that the variation of $\mathcal{K}$ can be written as

$$
\begin{aligned}
\mathcal{K}(\Omega\backslash\overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) \;=\;& \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)\mathbf{n}w_0^N ds - \nu\int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^N\right|^2 dx \\
& + \nu\int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^D\right|^2 dx - \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D, q_\varepsilon^D - q_0^D)\mathbf{n}w_0^D ds.
\end{aligned}
$$

4   Asymptotic behavior of the perturbed solutions

In this section, we discuss the influence of the geometric perturbation $\mathcal{O}_{z,\varepsilon}$ on the solutions of the Stokes problems $(\mathcal{P}_\varepsilon^N)$ and $(\mathcal{P}_\varepsilon^D)$. More precisely, we derive an asymptotic formula describing the variations of the velocity and pressure fields with respect to the perturbation size ε. We start our analysis by the perturbed Stokes Neumann problem.

4.1 The Neumann perturbed solution

This section is devoted to an asymptotic formula describing the variation of
the Neumann solution $(w_\varepsilon^N - w_0^N)$ with respect ε. We begin our analysis by
the following first order estimate.

Lemma 4.1. *Let $\mathcal{O}_{z,\varepsilon}$ be a small geometric perturbation strictly included into
Ω. Then, the perturbed Stokes solution $(w_\varepsilon^N, q_\varepsilon^N)$ satisfies the behavior*

$$w_\varepsilon^N(x) - w_0^N(x) = E_0^N((x-z)/\varepsilon) + O(\varepsilon) \ \text{in} \ \Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon},$$

$$q_\varepsilon^N(x) - q_0^N(x) = \frac{1}{\varepsilon}\Pi_0^N((x-z)/\varepsilon) + O(\varepsilon) \ \text{in} \ \Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}$$

*where the leading term (E_0^N, Π_0^N) is solution to the following Stokes exterior
problem*

$$\begin{cases} -\Delta E_0^N + \nabla\Pi_0^N &= 0 & in & \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ \nabla.E_0^N &= 0 & in & \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\ E_0^N &\longrightarrow 0 & at & \infty \\ E_0^N &= -w_0^N(z) & on & \partial\mathcal{O}. \end{cases} \tag{6}$$

Proof: The existence of (E_0^N, Π_0^N) can be established with the help of a single
layer potential [33]. One can derive

$$E_0^N(y) = \int_{\partial\mathcal{O}} U(y-t)\,\eta_0^N(t)ds(t), \forall y \in \mathbb{R}^3 \setminus \overline{\mathcal{O}},$$

$$\Pi_0^N(y) = \int_{\partial\mathcal{O}} P(y-t).\eta_0^N(t)ds(t), \forall y \in \mathbb{R}^3 \setminus \overline{\mathcal{O}},$$

where (U, P) is the fundamental solution of the Stokes system in $\mathbb{R}^3$

$$U(y) = \frac{1}{8\pi\nu r}\Big(I + e_r e_r^T\Big), \ P(y) = \frac{y}{4\pi r^3}, \quad \forall y \in \mathbb{R}^3,$$

with $r = \|y\|$, $e_r = \dfrac{y}{r}$ and e_r^T is the transposed vector of e_r.
Here η_0^N is the solution to the following boundary integral equation

$$\int_{\partial\mathcal{O}} U(y-t)\,\eta_0^N(t)ds(t) = -w_0^N(z), , \forall y \in \partial\mathcal{O}.$$

Setting

$$z_\varepsilon^N(x) = w_\varepsilon^N(x) - w_0^N(x) - E_0^N((x-z)/\varepsilon),$$

$$s_\varepsilon^N(x) = q_\varepsilon^N(x) - q_0^N(x) - \frac{1}{\varepsilon}\Pi_0^N((x-z)/\varepsilon).$$

As one can observe, $(z_\varepsilon^N, s_\varepsilon^N)$ satisfies the system

$$\begin{cases} -\nu \Delta z_\varepsilon^N + \nabla s_\varepsilon^N &= 0 & \text{in} & \Omega \backslash \overline{\mathcal{O}}_{z,\varepsilon}, \\ \text{div } z_\varepsilon^N &= 0 & \text{in} & \Omega \backslash \overline{\mathcal{O}}_{z,\varepsilon}, \\ \sigma(z_\varepsilon^N, s_\varepsilon^N)\mathbf{n} &= -\dfrac{1}{\varepsilon}\sigma(E_0^N, \Pi_0^N)((x-z)/\varepsilon)\mathbf{n} & \text{on} & \Sigma, \\ w_\varepsilon^N &= -E_0^N((x-z)/\varepsilon) & \text{on} & \Gamma, \\ w_\varepsilon^N &= -w_0^N + w_0^N(z) & \text{on} & \partial\mathcal{O}_{z,\varepsilon}. \end{cases}$$

Using the change of variable $x = z + \varepsilon y$ and the standard energy estimate for the Stokes problem, one can derive that there exists a constant $c > 0$, independent of ε, such that

$$\left\| z_\varepsilon^N \right\|_{H^1(\Omega \backslash \overline{\mathcal{O}}_{z,\varepsilon})} + \left\| s_\varepsilon^N \right\|_{L^2(\Omega \backslash \overline{\mathcal{O}}_{z,\varepsilon})} \le c\,\varepsilon.$$

One can see ([19], Proposition 3.1) for similar proof.

$\square$

Next, we extend this estimate to the high-order case. The obtained asymptotic behavior is illustrated by the following theorem.

Theorem 4.2. *Let $\mathcal{O}_{z,\varepsilon} = z + \varepsilon\omega$ be a small geometric perturbation, strictly embedded in the fluid flow domain Ω. Then, the velocity and pressure fields satisfy the following asymptotic behavior*

$$w_\varepsilon^N(x) - w_0^N(x) = E_0^N((x-z)/\varepsilon) + \sum_{k=1}^m \varepsilon^k [V_k^N(x) + E_k^N((x-z)/\varepsilon)] + o(\varepsilon^m), \quad (7)$$

$$q_\varepsilon^N(x) - q_0^N(x) = \frac{1}{\varepsilon}\Pi_0^N((x-z)/\varepsilon) + \sum_{k=1}^m \varepsilon^k [S_k^N(x) + \frac{1}{\varepsilon}\Pi_k^N((x-z)/\varepsilon))] + o(\varepsilon^m),$$
$$(8)$$

where $(V_k^N, S_k^N)_{0 \le k \le m}$ is a set of smooth functions satisfying the Stokes system in Ω and $(E_k^N, \Pi_k^N)_{0 \le k \le m}$ is a set of smooth functions verifying the exterior Stokes problem in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$.

Proof: The terms of the derived asymptotic expansion are built iteratively.

Initialization: We start our construction process by the terms associated with $k = 0$. The two sequences $(V_k^N, S_k^N)_{0 \le k \le m}$ and $(E_k^N, \Pi_k^N)_{0 \le k \le m}$ are initialized as follows:

$-$ $(V_0^N, S_0^N) = (w_0^N, q_0^N)$ which is the solution to the Stokes problem $(\mathcal{P}_0^N)$, defined in

the non perturbed domain Ω.

$-$ (E_0^N, Π_0^N) is the solution to the exterior Stokes problem (6), defined in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$.

The kth term : Let $k \in \{1, ..., m\}$. Assume that we have already derived the terms (V_i^N, S_i^N) and (E_i^N, Π_i^N) for all $0 \leq i \leq k-1$, and we want to derive the terms (V_k^N, S_k^N) and (E_k^N, Π_k^N).

In order to define the desired terms, we need to establish a preliminary calculus. It concerns the asymptotic behavior of the functions E_i^N and Π_i^N with respect to ε.

Recalling that (E_i^N, Π_i^N) is constructed as a solution to an exterior Stokes problem defined in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. Then, due to a single layer potential [33], (E_i^N, Π_i^N) can be written as

$$E_i^N(y) = \int_{\partial \mathcal{O}} U(y - t)\, \eta_i^N(t) ds(t), \, \forall y \in \mathbb{R}^3 \setminus \overline{\mathcal{O}},$$

$$\Pi_i^N(y) = \int_{\partial \mathcal{O}} P(y - t).\eta_i^N(t) ds(t), \, \forall y \in \mathbb{R}^3 \setminus \overline{\mathcal{O}},$$

where η_i^N is the solution to a boundary integral equation defined on $\partial \mathcal{O}$.

From the fact that $U(y/\varepsilon) = \varepsilon U(y)$ it follows that for each $x \in \mathbb{R}^3 \setminus \overline{\mathcal{O}_{z,\varepsilon}}$ we have

$$E_i^N((x-z)/\varepsilon) = \int_{\partial \mathcal{O}} U((x-z)/\varepsilon - t)\, \eta_i^N(t) ds(t)$$

$$= \varepsilon \int_{\partial \mathcal{O}} U((x-z) - \varepsilon\, t)\, \eta_i^N(t) ds(t).$$

From the fact that $\mathcal{O}_{z,\varepsilon}$ is not close to the boundary $\partial \Omega$, one can remark that for all $t \in \partial \mathcal{O}$, the function $\varepsilon \longmapsto U_{x-z,t}(\varepsilon) = \varepsilon\, U((x-z) - \varepsilon\, t)$ is smooth with respect to ε and admits the following asymptotic expansion

$$U_{x-z,t}(\varepsilon) = \sum_{p=1}^{m} \frac{\varepsilon^p}{(p-1)!} U_{x-z,t}^{(p-1)}(0) + o(\varepsilon^m),$$

where $U_{x-z,t}^{(p)}(0)$ is the pth derivative of $U_{x-z,t}$ at $\varepsilon = 0$. It depends on the pth derivative of the function U at the point $x - z$. Consequently, the function

$\varepsilon \longmapsto E_i^N((x-z)/\varepsilon)$ satisfies the following asymptotic behavior

$$E_i^N((x-z)/\varepsilon) = \sum_{p=1}^{m} \varepsilon^p \, E_{N,i}^{(p)}(x-z) + o(\varepsilon^m), \tag{9}$$

with $E_{N,i}^{(p)}$ is the smooth function defined in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$ by

$$E_{N,i}^{(p)}(y) = \frac{1}{(p-1)!} \int_{\partial\mathcal{O}} U_{y,t}^{(p-1)}(0)\eta_i^N(t)ds(t), \; \forall y \in \mathbb{R}^3 \setminus \overline{\mathcal{O}}. \tag{10}$$

Similarly, we applied the same analysis for the pressure field. Then, using the fact that $P(y/\varepsilon) = \varepsilon^2 P(y)$ one can obtain

$$\Pi_i^N((x-z)/\varepsilon) = \varepsilon^2 \int_{\partial\mathcal{O}} P((x-z) - \varepsilon\,t).\eta_i^N(t)ds(t).$$

From the smoothness of the function $\varepsilon \longmapsto P_{x-z,t}(\varepsilon) = \varepsilon^2 \, P((x-z) - \varepsilon\,t)$ with respect to ε, it follows

$$P_{x-z,t}(\varepsilon) = \sum_{q=1}^{m-1} \frac{\varepsilon^{q+1}}{(q-1)!} P_{x-z,t}^{(q-1)}(0) + o(\varepsilon^m),$$

where $P_{x-z,t}^{(q)}(0)$ is the qth derivative of $P_{x-z,t}$ at $\varepsilon = 0$. Hence, we have

$$\Pi_i^N((x-z)/\varepsilon) = \sum_{q=1}^{m-1} \varepsilon^{q+1} \, \Pi_{N,i}^{(q)}(x-z) + o(\varepsilon^m), \tag{11}$$

with $\Pi_{N,i}^{(q)}$ is the smooth function defined in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$ by

$$\Pi_{N,i}^{(q)}(y) = \frac{1}{(q-1)!} \int_{\partial\mathcal{O}} P_{y,t}^{(q-1)}(0).\eta_i^N(t)ds(t), \; \forall y \in \mathbb{R}^3 \setminus \overline{\mathcal{O}}. \tag{12}$$

$-Determining\ the\ term\ \ (V_k^N, S_k^N)$. It is constructed using the functions $E_i^N, 0 \le i \le k - 1$. It is defined as the solution to the following Stokes system

$$\begin{cases} -\Delta V_k^N + \nabla S_k^N &= 0 & \text{in} \quad \Omega, \\ \nabla.V_k^N &= 0 & \text{in} \quad \Omega, \\ \sigma(V_k^N, S_k^N)\mathbf{n} &= -\displaystyle\sum_{p=1}^{k} \sigma(E_{N,k-p}^{(p)}, \Pi_{N,k-p}^{(p)})(x-z)\mathbf{n} & \text{on} \quad \Sigma, \\ V_k^N &= -\displaystyle\sum_{p=1}^{k} E_{N,k-p}^{(p)}(x-z) & \text{on} \quad \Gamma, \end{cases} \tag{13}$$

where the functions $E_{N,j}^{(p)}$ and $\Pi_{N,j}^{(p)}$ are defined by (10) and (12).

$-Determining\ the\ term$ (E_k^N, Π_k^N). It is constructed using the functions V_i^N, $0 \le i \le k$. This term is defined as a solution to the following exterior problem

$$
\begin{cases}
-\Delta E_k^N + \nabla \Pi_k^N & = 0 & \text{in} & \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\
\nabla . E_k^N & = 0 & \text{in} & \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\
E_k^N & \longrightarrow 0 & \text{at} & \infty \\
E_k^N & = -V_k^N(z) - \displaystyle\sum_{p=1}^{k} \frac{1}{p!} D^p V_{k-p}^N(z)(y^p) & \text{on} & \partial \mathcal{O},
\end{cases}
\tag{14}
$$

where $D^p V_{k-p}^N(z)$ is the pth derivative of the function V_{k-p}^N at the point z and $y^p = (y, ..., y) \in (\mathbb{R}^3)^p$.

Justification of the asymptotic formulas: Here we will prove that the constructed sequences $(V_k^N, S_k^N)_{0 \le k \le m}$ and $(E_k^N, \Pi_k^N)_{0 \le k \le m}$ permit us to derive the expected asymptotic formulas.
Posing

$$
R_{m,\varepsilon}^N(x) = w_0^N(x) + E_0^N((x-z)/\varepsilon) + \sum_{k=1}^{m} \varepsilon^k [V_k^N(x) + E_k^N((x-z)/\varepsilon)] - w_\varepsilon^N,
$$

$$
\xi_{m,\varepsilon}^N(x) = S_0^N(x) + \frac{1}{\varepsilon} \Pi_0^N((x-z)/\varepsilon) + \sum_{k=1}^{m} \varepsilon^k [S_k^N(x) + \frac{1}{\varepsilon} \Pi_k^N((x-z)/\varepsilon)] - q_\varepsilon^N.
$$

One can easily verify that $(R_{m,\varepsilon}^N, \xi_{m,\varepsilon}^N)$ solves the Stokes system in $\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}$

$$
\begin{aligned}
-\Delta R_{m,\varepsilon}^N + \nabla \xi_{m,\varepsilon}^N & = 0 & \text{in } \Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}, \\
\nabla . R_{m,\varepsilon}^N & = 0 & \text{in } \Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon},
\end{aligned}
\tag{15}
$$

and satisfies the following boundaries conditions:

- On $\partial \mathcal{O}_{z,\varepsilon}$: Using the systems (13)-(14), the multi-linearity of $D^p V_{k-p}^N(z)$, Taylor's Theorem and the fact that $\|x-z\| = O(\varepsilon)$ on $\partial \mathcal{O}_{z,\varepsilon}$, one can derive

$$
R_{m,\varepsilon}^N(x) = \sum_{k=0}^{m} \varepsilon^k \left[V_k^N(x) - \sum_{p=0}^{m-k} \frac{1}{p!} D^p V_k^N(z)((x-z)^p) \right] = o(\varepsilon^m).
$$

- On Γ: the Dirichlet boundary condition in (13) and the asymptotic expansions (9) and (11) imply

$$
\begin{aligned}
R_{m,\varepsilon}^N(x) &= \sum_{k=0}^m \varepsilon^k E_k^N((x-z)/\varepsilon)) + \sum_{k=1}^m \varepsilon^k V_k^N(x) \\
&= \sum_{k=0}^m \varepsilon^k \Big[\sum_{p=1}^m \varepsilon^p E_{N,k}^{(p)}(x-z) \Big] - \sum_{k=1}^m \varepsilon^k \Big[\sum_{p=1}^k E_{N,k-p}^{(p)}(x-z) \Big] + o(\varepsilon^m) \\
&= o(\varepsilon^m).
\end{aligned}
\tag{16}
$$

- On Σ: by the change of variable $x = z + \varepsilon y$, we have

$$
\sigma(R_{m,\varepsilon}, \xi_{m,\varepsilon}) = \frac{1}{\varepsilon} \sum_{k=0}^m \varepsilon^k \sigma_y(E_k^N, \Pi_k^N)((x-z)/\varepsilon) + \sum_{k=1}^m \varepsilon^k \sigma(V_k^N, S_k^N)(x).
$$

Using again the change of variable, from (9) and (11) one can deduce

$$
\sigma_y(E_k^N, \Pi_k^N)((x-z)/\varepsilon) = \varepsilon \sum_{p=1}^m \varepsilon^p \sigma(E_{N,k}^{(p)}, \Pi_{N,k}^{(p)})(x-z) + o(\varepsilon^m).
$$

The two last relations combined with the Neumann condition used in (13) imply

$$
\sigma(R_{m,\varepsilon}, \xi_{m,\varepsilon}) \mathbf{n} = o(\varepsilon^m) \text{ on } \Sigma.
$$

4.2 The Dirichlet perturbed solution

This section is concerned with brief analysis of the Dirichlet case. Here, we consider $(V_k^D, S_k^D)_{0 \le k \le m}$ a set of smooth functions satisfying the Stokes system in Ω and $(E_k^D, \Pi_k^D)_{0 \le k \le m}$ a set of smooth functions verifying the exterior Stokes problem in $\mathbb{R}^3 \setminus \overline{\mathcal{O}}$. The two considered sequences $(V_k^D, S_k^D)_{0 \le k \le m}$ and $(E_k^D, \Pi_k^D)_{0 \le k \le m}$ are initialized as follows:

   — $(V_0^D, S_0^D) = (w_0^D, q_0^D)$ which is the solution to the Stokes problem $(\mathcal{P}_0^D)$.

   — (E_0^D, Π_0^D) is defined as the solution to the following exterior Stokes problem

$$
\begin{cases}
-\Delta E_0^D + \nabla \Pi_0^D &= 0 & \text{in} & \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\
\nabla . E_0^D &= 0 & \text{in} & \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\
E_0^D &\longrightarrow 0 & \text{at} & \infty \\
E_0^D &= -w_0^D(z) & \text{on} & \partial \mathcal{O}.
\end{cases}
$$

•*The term* (V_k^D, S_k^D). It is defined as the unique solution to the following Stokes system

$$
\begin{cases}
-\Delta V_k^D + \nabla S_k^D & = 0 & \text{in} & \Omega, \\
\nabla . V_k^D & = 0 & \text{in} & \Omega, \\
V_k^D & = -\displaystyle\sum_{p=1}^{k} E_{D,k-p}^{(p)}(x-z) & \text{on} & \Gamma \cup \Sigma,
\end{cases}
\tag{17}
$$

where the functions $E_{D,j}^{(p)}, 0 \le j \le k$ are defined by

$$
E_{D,j}^{(p)}(y) = \frac{1}{(p-1)!} \int_{\partial \mathcal{O}} U_{y,t}^{(p-1)}(0) \eta_j^D(t) ds(t), \ \forall y \in \mathbb{R}^3 \setminus \overline{\mathcal{O}}.
\tag{18}
$$

•*The term* (E_k^D, Π_k^D). It is constructed using the functions $V_j^D, 0 \le j \le k$. This term is defined as a solution to the following exterior problem

$$
\begin{cases}
-\Delta E_k^D + \nabla \Pi_k^D & = 0 & \text{in} & \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\
\nabla . E_k^D & = 0 & \text{in} & \mathbb{R}^3 \setminus \overline{\mathcal{O}}, \\
E_k^D & \longrightarrow 0 & \text{at} & \infty \\
E_k^D & = -V_k^D(z) - \displaystyle\sum_{p=1}^{k} \frac{1}{p!} D^p V_{k-p}^D(z)(y^p) & \text{on} & \partial \mathcal{O}.
\end{cases}
\tag{19}
$$

Theorem 4.3. *In the presence of a small geometric perturbation* $\mathcal{O}_{z,\varepsilon} = z + \varepsilon \omega$ *inside the fluid flow domain* Ω*, the solution* $(w_\varepsilon^D, q_\varepsilon^D)$ *of the perturbed Dirichlet Stokes problem admits the asymptotic behavior*

$$
w_\varepsilon^D(x) - w_0^D(x) = E_0^D((x-z)/\varepsilon) + \sum_{k=1}^{m} \varepsilon^k [V_k^D(x) + E_k^D((x-z)/\varepsilon)] + o(\varepsilon^m),
\tag{20}
$$

$$
q_\varepsilon^D(x) - q_0^D(x) = \frac{1}{\varepsilon} \Pi_0^D((x-z)/\varepsilon) + \sum_{k=1}^{m} \varepsilon^k [S_k^D(x) + \frac{1}{\varepsilon} \Pi_k^D((x-z)/\varepsilon)] + o(\varepsilon^m).
\tag{21}
$$

5   High-order topological asymptotic expansion

In this section we extend the topological derivative notion [16, 17, 18, 19, 20, 21, 34] for the high-order case. We derive a high-order terms in the topological asymptotic expansion for the Stokes operator. We will derive an asymptotic

formula describing the variation of the Kohn-Vogelius functional $\mathcal{K}$ with respect to the insertion of a small obstacle inside the fluid flow domain Ω.

From Theorem 3.1, the variation caused by the presence of the geometric perturbation $\mathcal{O}_{z,\varepsilon} = z + \varepsilon\mathcal{O}$ can be decomposed as

$$\mathcal{K}(\Omega \setminus \overline{\mathcal{O}}_{z,\varepsilon}) - \mathcal{K}(\Omega) \quad = \quad \mathcal{J}^N(\varepsilon) - \mathcal{J}^D(\varepsilon),$$

where the Neumann and Dirichlet terms are defined by

$$\mathcal{J}^N(\varepsilon) \quad = \quad \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)\mathbf{n}w_0^N ds - \nu \int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^N\right|^2 dx, \quad (22)$$

$$\mathcal{J}^D(\varepsilon) \quad = \quad \int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^D - w_0^D, q_\varepsilon^D - q_0^D)\mathbf{n}w_0^D ds - \nu \int_{\mathcal{O}_{z,\varepsilon}} \left|\nabla w_0^D\right|^2 dx. \quad (23)$$

To derive the expected high-order asymptotic expansion for Kohn-Vogelius functional $\mathcal{K}$ we will examine the terms $\mathcal{J}^N$ and $\mathcal{J}^D$ separately.

5.1   Estimate of the Neumann terms

Here, we derive a sensitivity analysis for each term in $\mathcal{J}^N(\varepsilon)$ with respect to the parameter ε. We will establish a high-order asymptotic expansion for each term. Our mathematical analysis is based on the asymptotic behavior of the perturbed solution $(w_\varepsilon^N, q_\varepsilon^N)$.

Lemma 5.1. *The first term in (22) admits the estimate*

$$\int_{\partial\mathcal{O}_{z,\varepsilon}} \sigma(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)\mathbf{n}.w_0^N ds$$

$$= \sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial\mathcal{O}} \sigma_y(E_k^N, \Pi_k^N)(y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y)$$

$$+ \sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial\mathcal{O}} \sigma(V_k^N, S_k^N)(z+\varepsilon y)\mathbf{n}.w_0^N(z+\varepsilon y)ds(y) + o(\varepsilon^m).$$

$$(24)$$

Proof: From the change of variable $x = z + \varepsilon y$, it follows

$$\nabla_x\left(E_k^N((x-z)/\varepsilon))\right) = \frac{1}{\varepsilon}\nabla_y E_k^N((x-z)/\varepsilon)), \quad 0 \leq k \leq m,$$

where ∇_ξ denotes the gradient with respect to the variable ξ ($\xi = x$ or y).

Using the previous relation and the decomposition presented in Theorem 4.2, one can derive

$$
\sigma(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N) = \frac{1}{\varepsilon}\sigma_y(E_0^N, \Pi_0^N)((x-z)/\varepsilon)) + \sum_{k=1}^{m}\varepsilon^k\sigma(V_k^N, S_k^N)(x)
$$

$$
+ \sum_{k=1}^{m+1}\varepsilon^{k-1}\sigma_y(E_k^N, \Pi_k^N)((x-z)/\varepsilon) + o(\varepsilon^m).
$$

Then, the first term in (22) satisfies the estimate

$$
\int_{\partial\mathcal{O}_{z,\varepsilon}}\sigma(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)\mathbf{n}.w_0^N ds = \frac{1}{\varepsilon}\int_{\partial\mathcal{O}_{z,\varepsilon}}\sigma_y(E_0^N, \Pi_0^N)((x-z)/\varepsilon))\mathbf{n}.w_0^N ds
$$

$$
+ \sum_{k=1}^{m+1}\varepsilon^{k-1}\int_{\partial\mathcal{O}_{z,\varepsilon}}\sigma_y(E_k^N, \Pi_k^N)((x-z)/\varepsilon)\mathbf{n}.w_0^N ds
$$

$$
+ \sum_{k=1}^{m}\varepsilon^k\int_{\partial\mathcal{O}_{z,\varepsilon}}\sigma(V_k^N, S_k^N)(x)\mathbf{n}.w_0^N ds + o(\varepsilon^m).
$$

Making use of the change of variable $x = z + \varepsilon y$, one can deduce

$$
\int_{\partial\mathcal{O}_{z,\varepsilon}}\sigma_y(E_k^N, \Pi_k^N)((x-z)/\varepsilon)\mathbf{n}.w_0^N ds = \varepsilon^2\int_{\partial\mathcal{O}}\sigma_y(E_k^N, \Pi_k^N)(y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y),
$$

$$
\int_{\partial\mathcal{O}_{z,\varepsilon}}\sigma(V_k^N, S_k^N)(x)\mathbf{n}.w_0^N ds = \varepsilon^2\int_{\partial\mathcal{O}}\sigma(V_k^N, S_k^N)(z+\varepsilon y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y).
$$

Consequently, we obtain

$$
\int_{\partial\mathcal{O}_{z,\varepsilon}}\sigma(w_\varepsilon^N - w_0^N, q_\varepsilon^N - q_0^N)\mathbf{n}.w_0^N ds = \sum_{k=0}^{m-1}\varepsilon^{k+1}\int_{\partial\mathcal{O}}\sigma_y(E_k^N, \Pi_k^N)(y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y)
$$

$$
+ \sum_{k=1}^{m-2}\varepsilon^{k+2}\int_{\partial\mathcal{O}}\sigma(V_k^N, S_k^N)(z+\varepsilon y)\mathbf{n}.w_0^N(z+\varepsilon y)ds(y) + o(\varepsilon^m).
$$

$\square$

Next, we will examine the two integral terms in the right hand side of (24).

Lemma 5.2. *We have*

$$
\sum_{k=0}^{m-1}\varepsilon^{k+1}\int_{\partial\mathcal{O}}\sigma_y(E_k^N, \Pi_k^N)(y)\mathbf{n}(y).w_0^N(z+\varepsilon y)ds(y) = \sum_{q=0}^{m-1}\varepsilon^{q+1}\mathcal{K}_q^{1,N}(z, \mathcal{O}) + o(\varepsilon^m),
$$

where the functions $z \longmapsto \mathcal{K}_q^{1,N}(z, \mathcal{O})$, $0 \leq q \leq m$ *are defined in* Ω *by*

$$\mathcal{K}_q^{1,N}(z, \mathcal{O}) = \sum_{p=0}^{q} \frac{1}{p!} \int_{\partial \mathcal{O}} \sigma_y(E_{q-p}^N, \Pi_{q-p}^N)(y)\mathbf{n}(y).[D^{(p)}w_0^N(z)(y^p)]ds(y)$$

with $D^{(p)}w_0^N(z)$ *denotes the* p^{th} *derivative of the function* w_0^N *at the point* $z \in \Omega$ *and* $y^p = (y, ..., y) \in (\mathbb{R}^3)^p$.

Proof: Due to the smoothness of the velocity field w_0^N, by Taylor's theorem one can derive

$$w_0^N(z + \varepsilon y) = \sum_{p=0}^{m} \frac{\varepsilon^p}{p!} D^{(p)}w_0^N(z)(y^p) + o(\varepsilon^m). \tag{25}$$

It follows

$$\sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial \mathcal{O}} \sigma_y(E_k^N, \Pi_k^N)(y)\mathbf{n}(y).w_0^N(z + \varepsilon y)ds(y)$$

$$= \sum_{k=0}^{m-1} \varepsilon^{k+1}\Big(\sum_{p=0}^{m} \frac{\varepsilon^p}{p!} \int_{\partial \mathcal{O}} \sigma_y(E_k^N, \Pi_k^N)(y)\mathbf{n}(y).[D^{(p)}w_0^N(z)(y^p)]ds(y)\Big) + o(\varepsilon^m)$$

$$= \sum_{q=0}^{m-1} \varepsilon^{q+1}\Big(\sum_{p=0}^{q} \frac{1}{p!} \int_{\partial \mathcal{O}} \sigma_y(E_{q-p}^N, \Pi_{q-p}^N)(y)\mathbf{n}(y).[D^{(p)}w_0^N(z)(y^p)]ds(y)\Big) + o(\varepsilon^m)$$

$$\square$$

Lemma 5.3. *The second integral term in the right hand side of (24) satisfies*

$$\sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial \mathcal{O}} \sigma(V_k^N, S_k^N)(z + \varepsilon y)\mathbf{n}.w_0^N(z + \varepsilon y)ds(y) = \sum_{q=1}^{m-2} \varepsilon^{q+2}\mathcal{K}_q^{2,N}(z, \mathcal{O}) + o(\varepsilon^m).$$

where the functions $z \longmapsto \mathcal{K}_q^{2,N}(z, \mathcal{O})$, $1 \leq q \leq m - 2$ *are defined in* Ω *by*

$$\mathcal{K}_q^{2,N}(z, \mathcal{O}) = \sum_{p=0}^{q-1}\sum_{l=0}^{p} \frac{1}{l!(p-l)!} \int_{\partial \mathcal{O}} \sigma^{(l)}(V_{q-p}^N, S_{q-p}^N)(z)\mathbf{n}.D^{(p-l)}w_0^N(z)(y^{p-l})ds(y)$$

Proof: Here we exploit the smoothness of the functions V_q^N and S_q^N. This follows from the fact that (V_q^N, S_q^N) is solution to the Stokes problem (13),

defined in a smooth domain Ω and verifying smooth boundary data on $\partial\Omega$.
Using Taylor's formula, one can derive

$$
S_k^N(z + \varepsilon y) \quad = \quad \sum_{p=0}^m \frac{\varepsilon^p}{p!} D^{(p)}\left(S_k^N\right)(z)(y^p) + o(\varepsilon^m),
$$

$$
\frac{\partial(V_k^N)_i}{\partial x_j}(z + \varepsilon y) \quad = \quad \sum_{p=0}^m \frac{\varepsilon^p}{p!} D^{(p)}\left(\frac{\partial(V_k^N)_i}{\partial x_j}\right)(z)(y^p) + o(\varepsilon^m), \, 1 \le i, j \le 3.
$$

Recalling that

$$
\sigma(V_k^N, S_k^N)_{i,j} = \mu\left(\frac{\partial(V_k^N)_i}{\partial x_j} + \frac{\partial(V_k^N)_j}{\partial x_i}\right) - S_k^N \delta_{i,j}
$$

where $\delta_{i,j}$ is the Kronecker symbol.
Then, for each $1 \le i, j \le 3$, one can derive

$$
\sigma(V_k^N, S_k^N)_{i,j}(z + \varepsilon y) = \sum_{p=0}^m \frac{\varepsilon^p}{p!} \sigma_{i,j}^{(p)}(V_k^N, S_k^N)(z) + o(\varepsilon^m), \tag{26}
$$

with $\sigma_{i,j}^{(p)}(V_k^N, S_k^N)(z) = D^{(p)}\left(\sigma(V_k^N, S_k^N)_{i,j}\right)(z)$ which is the pth derivative
of the function $\mu\left(\dfrac{\partial(V_k^N)_i}{\partial x_j} + \dfrac{\partial(V_k^N)_j}{\partial x_i}\right) - S_k^N \delta_{i,j}$ at the point $z \in \Omega$.
Due to the Cauchy product formula, the relations (25) and (26) imply

$$
\int_{\partial\mathcal{O}} \sigma(V_k^N, S_k^N)(z + \varepsilon y)\mathbf{n}.w_0^N(z + \varepsilon y)ds(y) =
$$

$$
\sum_{p=0}^m \varepsilon^p \Big(\sum_{l=0}^p \frac{1}{l!(p-l)!} \int_{\partial\mathcal{O}} \sigma^{(l)}(V_k^N, S_k^N)(z)\mathbf{n}.D^{(p-l)}w_0^N(z)(y^{p-l})ds(y)\Big) + o(\varepsilon^m).
$$

Finally, we obtain

$$
\sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial\mathcal{O}} \sigma(V_k^N, S_k^N)(z + \varepsilon y)\mathbf{n}.w_0^N(z + \varepsilon y)ds(y) =
$$

$$
\sum_{q=1}^{m-2} \varepsilon^{q+2} \Big(\sum_{p=0}^{q-1} \sum_{l=0}^p \frac{1}{l!(p-l)!} \int_{\partial\mathcal{O}} \sigma^{(l)}(V_{q-p}^N, S_{q-p}^N)(z)\mathbf{n}.D^{(p-l)}w_0^N(z)(y^{p-l})ds(y)\Big) + o(\varepsilon^m).
$$

$\square$

Lemma 5.4. *The second term in (22) admits the estimate*

$$
\nu \int_{\mathcal{O}_{z,\varepsilon}} |\nabla w_0^N|^2 dx = \sum_{q=0}^{m-3} \varepsilon^{q+3} \mathcal{K}_q^{3,N}(z, \mathcal{O}) + o(\varepsilon^m),
$$

*where $\{z \longmapsto \mathcal{K}_q^{3,N}(z, \mathcal{O}), 0 \leq q \leq m-3\}$ is a set of functions, defined in Ω
by*

$$\mathcal{K}_q^{3,N}(z, \mathcal{O}) = \sum_{p=0}^{q} \frac{\nu}{p!(q-p)!} \int_{\mathcal{O}} D^{(p+1)} w_0^N(z)(y^p) : D^{(q-p+1)} w_0^N(z)(y^{q-p}) dy.$$

Proof: The change of variable $x = z + \varepsilon y$, implies

$$\int_{\mathcal{O}_{z,\varepsilon}} |\nabla w_0^N|^2 dx = \varepsilon^3 \int_{\mathcal{O}} |\nabla w_0^N(z + \varepsilon y)|^2 dy.$$

Since w_0^N is smooth near z, by Taylor's theorem it follows

$$\nabla w_0^N(z + \varepsilon y) = \sum_{p=0}^{m} \frac{1}{p!} D^{(p+1)} w_0^N(z)(y^p) + o(\varepsilon^m).$$

By Cauchy product formula, we obtain

$$|\nabla w_0^N|^2(z+\varepsilon y) = \sum_{q=0}^{m} \varepsilon^q \left(\sum_{p=0}^{q} \frac{1}{p!(q-p)!} D^{(p+1)} w_0^N(z)(y^p) : D^{(q-p+1)} w_0^N(z)(y^{q-p}) \right) + o(\varepsilon^m).$$

$$\square$$

5.2   Estimate of the Dirichlet terms

In this section, we examine the Dirichlet terms involved in the Kohn-Vogelius
functional variation. Based on the asymptotic behavior of the perturbed so-
lution $(w_\varepsilon^D, q_\varepsilon^D)$ with respect to the parameter ε, we establish a high-order
asymptotic formula for each term of the function $\mathcal{J}^D(\varepsilon)$.
Using Theorem 4.3, the function $\mathcal{J}^D(\varepsilon)$ admits the following estimate

$$\begin{aligned}
\mathcal{J}^D(\varepsilon) = \quad & \sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial \mathcal{O}} \sigma_y(E_k^D, \Pi_k^D)(y)\mathbf{n}(y).w_0^D(z + \varepsilon y) ds(y) \\
& + \sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial \mathcal{O}} \sigma(V_k^D, S_k^D)(z + \varepsilon y)\mathbf{n}.w_0^D(z + \varepsilon y) ds(y) \\
& - \varepsilon^3 \nu \int_{\mathcal{O}} |\nabla w_0^D(z + \varepsilon y)|^2 dy. + o(\varepsilon^m).
\end{aligned}$$

Similar to the Naumann case, we derive the following preliminary lemmas
estimating the integral terms in the last equality.

Lemma 5.5. *We have*

$$\sum_{k=0}^{m-1} \varepsilon^{k+1} \int_{\partial \mathcal{O}} \sigma_y(E_k^D, \Pi_k^D)(y)\mathbf{n}(y).w_0^D(z+\varepsilon y)ds(y) = \sum_{q=0}^{m-1} \varepsilon^{q+1}\mathcal{K}_q^{1,D}(z,\mathcal{O}) + o(\varepsilon^m),$$

where the functions $z \longmapsto \mathcal{K}_q^{1,D}(z,\mathcal{O}),\ 0 \le q \le m$ *are defined in* Ω *by*

$$\mathcal{K}_q^{1,D}(z,\mathcal{O}) = \sum_{p=0}^{q} \frac{1}{p!} \int_{\partial \mathcal{O}} \sigma_y(E_{q-p}^D, \Pi_{q-p}^D)(y)\mathbf{n}(y).[\nabla^{(p)}w_0^D(z)(y^p)]ds(y)$$

Lemma 5.6. *The second integral term in the right hand side of (24) satisfies*

$$\sum_{k=1}^{m-2} \varepsilon^{k+2} \int_{\partial \mathcal{O}} \sigma(V_k^D, S_k^D)(z+\varepsilon y)\mathbf{n}.w_0^D(z+\varepsilon y)ds(y) = \sum_{q=1}^{m-2} \varepsilon^{q+2}\mathcal{K}_q^{2,D}(z,\mathcal{O}) + o(\varepsilon^m),$$

where the functions $z \longmapsto \mathcal{K}_q^{2,D}(z,\mathcal{O}),\ 1 \le q \le m-2$ *are defined in* Ω *by*

$$\mathcal{K}_q^{2,D}(z,\mathcal{O}) = \sum_{p=0}^{q-1}\sum_{l=0}^{p} \frac{1}{l!(p-l)!} \int_{\partial \mathcal{O}} \sigma^{(l)}(V_{q-p}^D, S_{q-p}^D)(z)\mathbf{n}.\nabla^{(p-l)}w_0^D(z)(y^{p-l})ds(y)$$

Lemma 5.7. *The second term in (22) admits the estimate*

$$\nu \int_{\mathcal{O}} |\nabla w_0^D(z+\varepsilon y)|^2 dy = \sum_{q=0}^{m} \varepsilon^q \mathcal{K}_q^{3,D}(z,\mathcal{O}) + o(\varepsilon^m),$$

where $\{z \longmapsto \mathcal{K}_q^{3,D}(z,\mathcal{O}),\ 0 \le q \le m\}$ *is a set of functions, defined in* Ω *by*

$$\mathcal{K}_q^{3,D}(z,\mathcal{O}) = \sum_{p=0}^{q} \frac{1}{p!(q-p)!} \int_{\mathcal{O}} \nabla^{(p+1)}w_0^D(z)(y^p) : \nabla^{(q-p+1)}w_0^D(z)(y^{q-p})dy.$$

5.3   Topological asymptotic formula

In this section, we derive a high-order topological asymptotic expansion for
the Kohn-Vogelius functional $\mathcal{K}$. The main result of this section is described
by Theorem 5.8.

Theorem 5.8. *Let* $\mathcal{O}_{z,\varepsilon} = z+\varepsilon\mathcal{O}$ *be a small obstacle inserted in the fluid flow
domain* Ω. *In the presence of* $\mathcal{O}_{z,\varepsilon}$, *the Kohn-Vogelius functional* $\mathcal{K}$ *satisfies
the following high-order asymptotic expansion*

$$\mathcal{K}(\Omega\backslash\overline{\mathcal{O}_{z,\varepsilon}}) = \mathcal{K}(\Omega) + \sum_{q=1}^{m} \varepsilon^q \delta^q \mathcal{K}(z,\mathcal{O}) + o(\varepsilon^m),$$

where $\delta^q \mathcal{K}$ is the qth topological derivative order, defined in Ω by:

- *If $q = 1, 2$*

$$\delta^q \mathcal{K}(z, \mathcal{O}) = \left[\mathcal{K}_{q-1}^{1,N} - \mathcal{K}_{q-1}^{1,D} \right] (z, \mathcal{O})$$

- *If $3 \leq q \leq m$*

$$\delta^q \mathcal{K}(z, \mathcal{O}) = \left[\mathcal{K}_{q-1}^{1,N} - \mathcal{K}_{q-1}^{1,D} \right] (z, \mathcal{O}) + \left[\mathcal{K}_{q-2}^{2,N} - \mathcal{K}_{q-2}^{2,D} \right] (z, \mathcal{O}) - \left[\mathcal{K}_{q-3}^{3,N} - \mathcal{K}_{q-3}^{3,D} \right] (z, \mathcal{O}).$$

Proof. According to Lemmas 5.1, 5.2, 5.3 and 5.4, the Neuman term $\mathcal{J}^N(\varepsilon)$ satisfies the estimate

$$\mathcal{J}^N(\varepsilon) = \sum_{q=0}^{m-1} \varepsilon^{q+1} \mathcal{K}_q^{1,N}(z, \mathcal{O}) + \sum_{q=1}^{m-2} \varepsilon^{q+2} \mathcal{K}_q^{2,N}(z, \mathcal{O}) - \sum_{q=0}^{m-3} \varepsilon^{q+3} \mathcal{K}_q^{3,N}(z, \mathcal{O}) + o(\varepsilon^m).$$

Based on Lemmas 5.5, 5.6 and 5.7, the Dirichlet term $\mathcal{J}^D(\varepsilon)$ can be estimated as

$$\mathcal{J}^D(\varepsilon) = \sum_{q=0}^{m-1} \varepsilon^{q+1} \mathcal{K}_q^{1,D}(z, \mathcal{O}) + \sum_{q=1}^{m-2} \varepsilon^{q+2} \mathcal{K}_q^{2,D}(z, \mathcal{O}) - \sum_{q=0}^{m-3} \varepsilon^{q+3} \mathcal{K}_q^{3,D}(z, \mathcal{O}) + o(\varepsilon^m).$$

Combining the two previous estimates and using the fact that

$$\mathcal{K}(\Omega \backslash \overline{\mathcal{O}_{z,\varepsilon}}) - \mathcal{K}(\Omega) = J^N(\varepsilon) - \mathcal{J}^D(\varepsilon),$$

one can derive the desired asymptotic formula.

$\square$

6   Conclusion

In this paper, we have derived a high-order topological asymptotic formula describing the variation of the Kohn-Vogelius functional with respect to the presence of a small obstacle immersed in the fluid flow domain. The obtained formula can serve as very useful tools for the numerical identification of the location "z", the size "ε" and the shape "$\mathcal{O}$" of the unknown obstacle.

From the asymptotic formula in Theorem 5.8, it now follows that, up to terms of smaller order, the unknown parameters z, ε and $\mathcal{O}$ can be characterized as the solution of a parameters estimate problem minimizing the nonlinear scalar function

$$\mathcal{F}(\varepsilon, z, \mathcal{O}) = \sum_{q=1}^{m} \varepsilon^q \delta^q \mathcal{K}(z, \mathcal{O}).$$

A first task of the identification process, is then the determination of the location "z" (center of the obstacle) and the size"ε". A second task would be (as well as possible) the determination of other information about the obstacle, such as its shape "$\mathcal{O}$".

A detailed account of this work and some numerical investigations will be the subject of a forthcoming paper.

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